Recall: Introduce Levi-Civita connection on Riemannian mfd (M, g) $\boxed{\nabla :}$

① $$Z[g(X,Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

$$\forall\ X, Y, Z \in \Gamma(TM)$$

② $$\nabla_X Y - \nabla_Y X = [X, Y] \quad \text{(torsion free)}$$

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k \quad \text{where} \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

Extension to tensor field:

① For $X \in \Gamma(TM)$, Define $\nabla_X : C^\infty(\underline{\otimes^s TM}) \to C^\infty(\otimes^s TM)$

by $$\nabla_X (Z_1 \otimes \cdots \otimes Z_s) = \sum_{i=1}^{s} Z_1 \otimes \cdots \otimes \nabla_X Z_i \otimes \cdots \otimes Z_s$$

② If α is (r, s) tensor, then define

$$(\nabla_X \alpha) = (r, s) \text{ tensor.}$$

by $$(\nabla_X \alpha)(Y_1, \cdots, Y_r) = \nabla_X(\underbrace{\alpha(Y_1, \cdots, Y_r)}_{(0,s)\ \text{tensor}}) - \sum_{i=1}^{r} \alpha(Y_1, \cdots, \nabla_X Y_i, \cdots, Y_r)$$

Example Ⓐ If α = 1-form ((1,0) tensor, Eg $\alpha = \sum_{i=1}^{n} f_i \, dx^i$)

then $$(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y) \in C^\infty(M)$$

for each given $X, Y \in \Gamma(TM)$.

Ⓑ g = (2,0) tensor $= \sum_{i,j} g_{ij} \, dx^i \otimes dx^j$

Recall: $$0 = X(g(Y,Z)) - [g(\nabla_X Y, Z) + g(Y, \nabla_X Z)]$$

$$= (\nabla_X g)(Y, Z).$$

$$\Rightarrow \boxed{\nabla_X g = 0 \quad \forall X \in \Gamma(TM)}$$

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③ Define $\nabla : C^\infty(\otimes^{r,s} M) \longrightarrow C^\infty(\otimes^{r+1,s} M)$ by

$$(\nabla\alpha)(X, Y_1, \dots, Y_r) = (\nabla_X \alpha)(Y_1, \dots, Y_r).$$

"∇ is compatible with $g \Leftrightarrow \nabla g = 0$"

Inductively, may define Hessian of a tensor, given by

$$(\nabla^2 \alpha)(X, Y, Z, W)$$ (α := (2,0) tensor in this case)

$$= (\nabla_X \nabla \alpha)(Y, Z, W) = (\nabla_X \beta)(Y, Z, W)$$ ($\beta = \nabla\alpha$)

$$= X(\beta(Y, Z, W)) - \underline{\beta[\nabla_X Y, Z, W) - \beta(Y, \nabla_X Z, W) - \beta(Y, Z, \nabla_X W)}$$

$$= X\left((\nabla_X \alpha)(Z,W)\right) - (\nabla_{\nabla_X Y} \overset{\downarrow}{\alpha})(Z,W) - (\nabla_Y \alpha)(D_X Z, W) - (\nabla_Y \alpha)(Z, \nabla_X W)$$

$= \cdots$

\#

Inductively, we can define $\nabla^k \alpha$ of a tensor α.

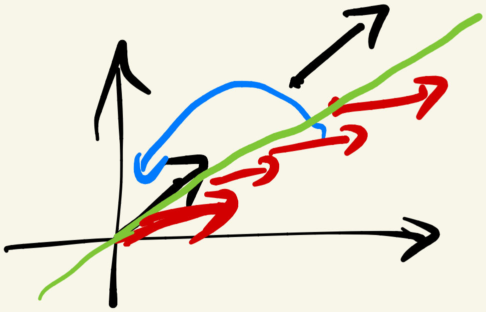
☆ Connection on $\Gamma(TM)$

$\rightsquigarrow$ induce

Connection on tensor field.

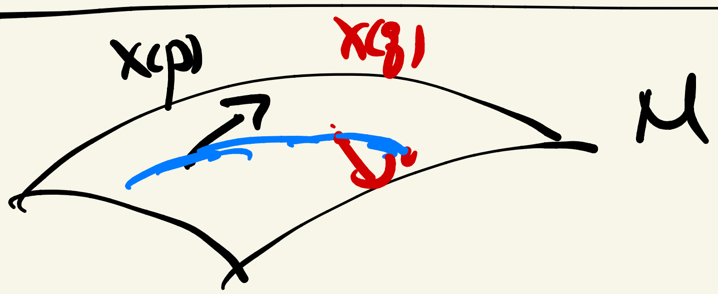
Goal: Differentiate Vector field on M.

In $\mathbb{R}^n$



$DV = 0$ iff

Vector field = constant



$X(p) \in T_pM$

$X(q) \in T_qM$

$\not\Rightarrow$

Defn: Given $\gamma : [0,1] \to M$, $\gamma(0) = p$, $\gamma(1) = q$ smooth curve. $\forall\, V_0 \in T_pM$, $\exists!$ $V(t)$ along $\gamma(t)$ s.t. $V(0) = V_0$, $\nabla_{\gamma'(t)} V(t) = 0$.

$$\boxed{V(1) = \underline{\text{parallel transport of}} \; \underline{V_0 \text{ along } \gamma}.}$$

Q: why exists ??

pf: let $\{E_i(t)\}_{i=1}^{n}$ be a smooth orthonormal basis for $T_{\gamma(t)}M$. (Done by Gram-Schmidt process)

$$\nabla_{\gamma'} V = 0 \iff \nabla_{\gamma'} \left(\langle V, E_i \rangle E_i \right) = 0$$

$$\|$$

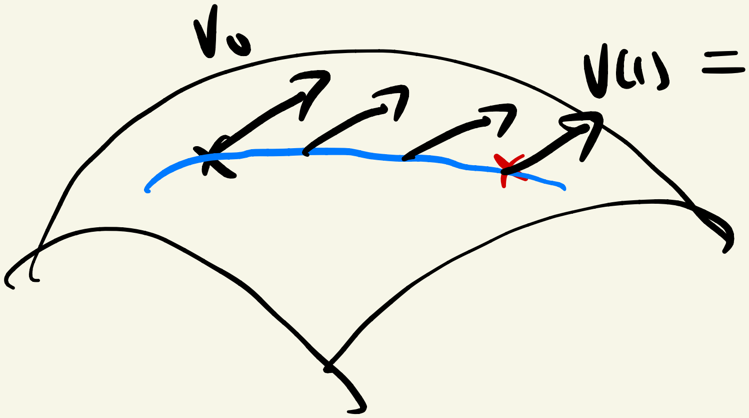
$$\gamma' \langle V, E_i \rangle E_i + \langle V, E_i \rangle \nabla_{\gamma'} E_i = 0$$

$$\underline{\underline{[\partial_t \langle V, E_i \rangle]}} E_i + \underline{\underline{\langle V, E_k \rangle}} \underbrace{\langle \nabla_{\gamma'} E_k, E_i \rangle}_{\text{fixed fcn.}} E_i = 0$$

$$\wr$$

linear ODE system with $V(0) = V_0$

By ODE theory, $V(t)$ exists and unique. #



$V(1)$ = parallel transport of V_0 along Blue curve.

Rmk: may Define

$$P_\gamma : T_{\gamma(0)}M \longrightarrow P_{\gamma(1)}M \text{ by}$$

$$\cdot\ P_\gamma V_0 = V(1)$$

Since

$$\gamma' \langle V(t), W(t) \rangle = \langle \nabla_{\gamma'} V, W \rangle + \langle V, \nabla_{\gamma'} W \rangle = 0$$

$$\forall\ V, W \text{ w/ } \nabla_{\gamma'} V = \nabla_{\gamma'} W = 0.$$

$\Rightarrow$ P_γ = isometry ☆☆.

<u>Locally</u> on a chart

$$g = g_{ij}\, dx^i \otimes dx^j$$

$$= \text{matrix locally}$$

$[g_{ij}]$: measure distance in infinitesmal scale

Goal: use g to define metric space structure on M.

given a curve: $\gamma : [a.b] \to M$. (smooth)

· $L(\gamma) = \int_a^b \sqrt{g(\gamma', \gamma')}\, dt.$

· Define $d(p,q)$ for $p, q \in M$ by

$$\|$$

$$\inf \{ L(\gamma) \mid \gamma : p \text{ to } q \}$$

Q: · Is (M, d) a metric space ??

· Is $(M, d) \cong M$ with original topology ??

· $\exists \gamma$ s.t. $L(\gamma) = d(p,q)$??

Ans: Yes !! // geodesic (minimizing)

Q1: · $d(p,q) \geq 0$ by defn.

· triangle ineq holds

suffices to check if $p \neq q$ then

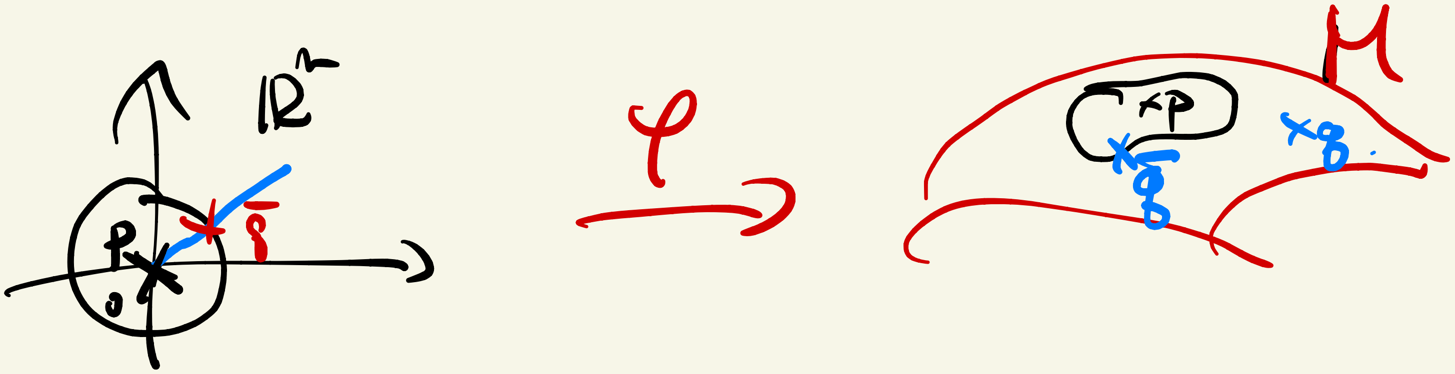
$d(p,q) > 0$.

pf: pick $r << 1$, s.t.

$\exists\ \varphi : B(r) \in \mathbb{R}^n \to M$ with $\varphi(0) = p$

$q \notin \varphi(B(r))$ and $\lambda^{-1} \delta_{ij} \leq g_{ij} \leq \lambda \delta_{ij}$

where δ_{ij} = Euclidean metric.



$\forall\ \bar{q} \in \partial(\varphi(B(r)))$,

let γ be curve from p to $\bar{q}$.

$\gamma : [a,b] \to M$ in $\varphi(B(r))$.

$$L(\gamma) = \int_a^b \sqrt{g(\gamma',\gamma')}\, dt$$

$$\geq \lambda^{-\frac{1}{2}} \int_a^b \sqrt{\underset{\sim}{\delta}(\gamma',\gamma')}\, dt$$

"Euclidean metric"

$$\geq \lambda^{-\frac{1}{2}} \cdot L_{\mathbb{R}^n}(\gamma)$$

$$= \lambda^{-\frac{1}{2}} \cdot r, \quad \boxed{\because \varphi^{-1}(\bar{q}) \in \partial B(r)}$$

$$\Rightarrow \quad d(p,\bar{q}) \geq \lambda^{-\frac{1}{2}} r > 0$$

$$\because q \notin \varphi(B(r)) \quad \therefore d(p,q) \geq \lambda^{-\frac{1}{2}} r > 0$$

This proved ①.

The proof also implies all small geodesic Ball comparable to Eucli. Ball.

$\forall\ q \to p$.

$$\lambda^{-\frac{1}{2}} d_{\mathbb{R}^n}(p,q) \le d(p,q) \le \lambda^{\frac{1}{2}} d_{\mathbb{R}^n}(p,q)$$

$$B_{\mathbb{R}^n}(p,r) \approx B_d(p,r) \text{ if } r \ll 1.$$

⇒ topology are same!!

proved ②.

Q3 : what about the minimizer??

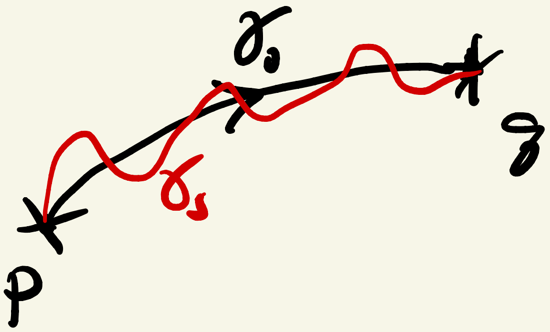
properties of γ (if exists):

$$\boxed{\underline{L(\gamma)} \le L(\tilde{\gamma}),\ \forall\ \tilde{\gamma} : p \text{ to } q.}$$

soln to minimization problem!! → Information to DE

⇓

$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = 0 \quad \forall\ \gamma_s$ = variation of γ.



- For simplicity, reparametrize γ_0 s.t. $|\gamma_0'| = 1$.

- $\gamma(s,t) : (-\varepsilon, \varepsilon) \times [a,b] \to M$ be s.t.

$$\begin{cases} \gamma(0,t) = \gamma_0(t) \\ \gamma(s,a) = p \\ \gamma(s,b) = q. \end{cases}$$

} fix end pts.

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{d}{ds}\Big|_{s=0} \int_a^b \sqrt{g(\gamma_s'(t), \gamma_s'(t))}\, dt$$

(where $\gamma_s'(t) = \partial_t \gamma(s,t)$)

$$= \int_a^b \frac{1}{2\cancelto{1}{|\gamma_0'|}} \partial_s \langle \gamma_s', \gamma_s' \rangle \, dt \Big|_{s=0}.$$

$$= \int_a^b \langle \nabla_{\partial_s} \gamma', \gamma' \rangle dt \Big|_{s=0}$$

$$= \int_a^b \langle \nabla_{\gamma'} \partial_s, \gamma' \rangle dt \Big|_{s=0}$$

$$= \int_a^b \frac{d}{dt} \langle \partial_s, \partial_t \rangle - \langle \partial_s, \nabla_{\gamma'} \gamma' \rangle dt \Big|_{s=0}$$

$$= \cancelto{0}{\langle \partial_s, \partial_t \rangle \Big|_a^b} - \int_a^b \langle \partial_s, \nabla_{\gamma'} \gamma' \rangle dt \Big|_{s=0}$$

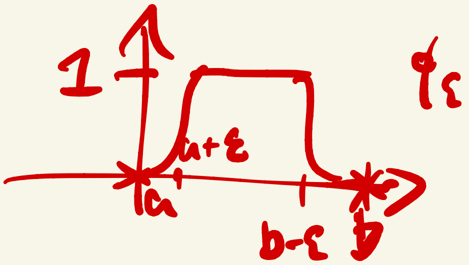
$$\boxed{\begin{aligned} &\because \ \gamma(s,a) = p, \quad \gamma(s,b) = q. \\ &\Rightarrow \ \partial s \big|_p = \partial s \big|_q = 0. \end{aligned}}$$

$$= - \int_a^b \langle V, \nabla_{\gamma'} \gamma' \rangle dt = 0.$$

where V = push forward of ∂s.

with $\underline{V(a) = 0}$, $\underline{V(b) = 0}$. #

taking $V(t) = \phi_\varepsilon(t) \nabla_{\gamma'}\gamma'\big|_{\gamma(t)}$



$$\Downarrow$$

$$\nabla_{\gamma'}\gamma' = 0 \quad \text{on} \quad [a+\varepsilon, b-\varepsilon], \ \forall \varepsilon > 0.$$

$$\Rightarrow \quad \nabla_{\gamma'}\gamma' \equiv 0 \quad \text{on} \quad [a,b] \quad \text{if} \quad \nabla_{\gamma'}\gamma' \in C^\infty.$$

Defn: We say that $\gamma : [a,b] \to M$ is a geodesic if $\nabla_{\gamma'}\gamma' = 0$ on $[a,b]$.

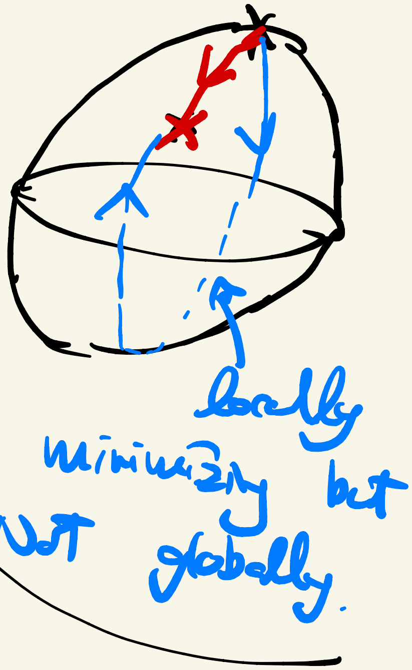
★ $\nabla_{\gamma'}\gamma' = 0 \Rightarrow \gamma'\langle\gamma', \gamma'\rangle = 0$

$\Rightarrow |\gamma'| =$ constant along γ.

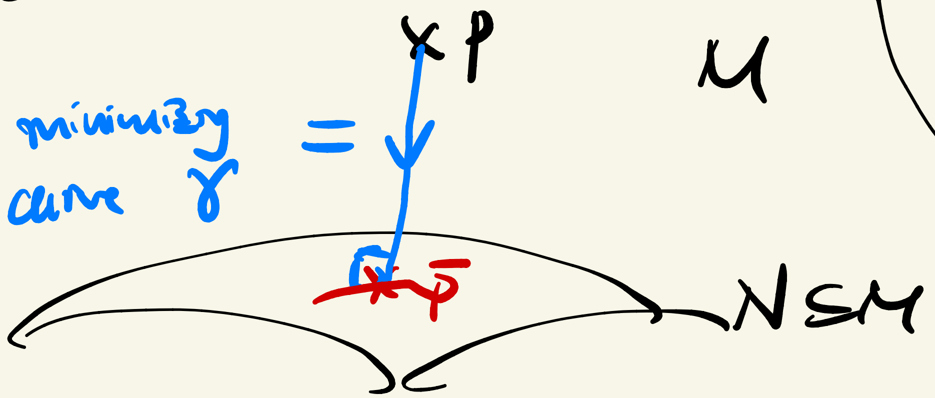
Usually, if $|\gamma'| = 1$, then called γ to be normal geodesic.

The discussion above $\Rightarrow$ C^∞ geodesic are (local) length minimizing.

Example of sphere



picture :



$$d(p,N) = \inf \{ L(\gamma) \mid \gamma : p \text{ to } N \}$$

prop : let N be sub-mfd w/o boundary.

$\gamma : [a,b] \to M$ be s.t. $\gamma(b) \in N$, $\gamma(a) = p$

$L(\gamma) = d(p,N)$, then $\gamma'(1) \perp T_{p}N$.

pf : Consider $\gamma_s : (-\varepsilon, \varepsilon) \times [a,b] \to M$

with $\gamma_0 = \gamma$, $\gamma_s(a) = p$, $\gamma_s(b) \in N$.

$$\boxed{\text{Normalize s.t. } |\gamma_0'| = 1}$$

$$\Rightarrow \frac{d}{ds}\Big|_{s=0} L(\gamma_s) = 0$$

$$\|$$

$$\langle V, \gamma' \rangle \Big|_a^b - \int_a^b \langle V, \cancel{\nabla_{\gamma'} \gamma'} \rangle \, dt$$

$d(p,\bar{p}) = L(\gamma_0) \Rightarrow \gamma_0$ = geodesic from p to $\bar{p}$.

$$\Rightarrow \nabla_{\gamma'}\gamma' = 0 .$$

$$\Rightarrow \langle V, \gamma' \rangle \Big|_a^b = 0 , \quad \forall V \text{ s.t. } \begin{cases} V(a) = 0 \\ V(b) \in T_{\bar{p}}N . \end{cases}$$

variational vector field

$$\Rightarrow \langle V, \gamma' \rangle (\bar{p}) = 0 \quad \forall V(\bar{p}) \in T_{\bar{p}}N .$$

$$\Rightarrow \boxed{\gamma'(b) \perp T_{\bar{p}}N .}$$

$[T, V] = 0$ because Rectangle

$$\gamma(s,t) : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M$$

$$T = d\gamma(\partial_t) , \quad V = d\gamma(\partial_s)$$

$$\Rightarrow [T, V] = [d\gamma(\partial_t), d\gamma(\partial_s)] = \underset{\overset{\shortparallel}{0}}{[\partial_t, \partial_s]}$$

Q: Does geodesic exists ??

i.e. $\exists$? $\gamma : [a,b] \to M$ with $\nabla_{\gamma'}\gamma' = 0$, $\gamma(a) = p$

$\gamma(b) = q$, $\gamma'(0) = v$??

for a given initial vector $v \in T_pM$??

Locally, $\gamma(t) = (x^1(t), x^2(t), \dots, x^n(t))$

$$\gamma'(t) = \sum_{i=1}^{n} \frac{dx^i}{dt} \cdot \frac{\partial}{\partial x^i} \in T_{\gamma(t)}M.$$

$$0 = \nabla_{\gamma'}\gamma' = \nabla_{\gamma'}\left(\frac{dx^i}{dt} \cdot \frac{\partial}{\partial x^i}\right)$$

$$= \frac{d^2x^i}{dt^2} \cdot \frac{\partial}{\partial x^i} + \frac{dx^i}{dt} \nabla_{\frac{dx^j}{dt}\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$$

$$= \frac{d^2x^i}{dt^2} \cdot \frac{\partial}{\partial x^i} + \frac{dx^i}{dt} \frac{dx^j}{dt} \left(\nabla_{\partial_j} \partial_i\right)$$

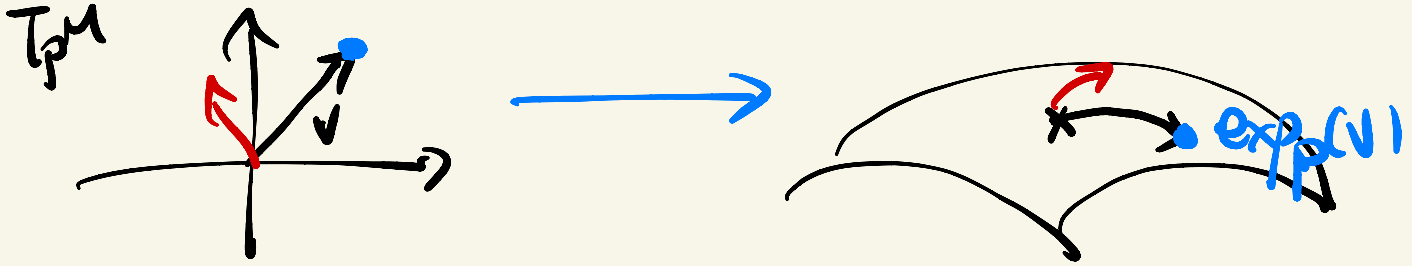
$$= \frac{d^2x^i}{dt^2} \frac{\partial}{\partial x^i} + \frac{dx^k}{dt} \cdot \frac{dx^j}{dt} \Gamma^i_{kj} \frac{\partial}{\partial x^i}$$

$$\boxed{0 = \left(\frac{d^2x^i}{dt^2} + \sum_{jk}\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\right), \forall i.}$$

local expression of $\boxed{\nabla_{\gamma'}\gamma' = 0.}$

2nd order ODE. $\xRightarrow{\text{ODE}}$ Existence and uniqueness for short time. for given initial data.

$$\boxed{\begin{array}{l}\therefore\ \exists!\ \gamma : (-\varepsilon, \varepsilon) \to M.\ \text{s.t.} \\ \quad \nabla_{\gamma'}\gamma' = 0 \text{ and } \gamma(0) = p,\ \gamma'(0) = V_0 \in T_pM. \\ \text{(local Existence of geodesic from } p \in M.)\end{array}}$$



Define: $\exp_p$: "T_pM" $\to M$. by

$\exp_p(V) = \gamma_V(1)$ where γ_V = geodesic from

$p = \gamma_V(0)$, $\gamma_V'(0) = V$ and γ_V is defined up to $t=1$..

---

observe: ODE $\Longrightarrow \forall v \in T_pM$, γ_v = geodesic
with $\gamma_v'(0)=v$ must exists
for $|t| < \varepsilon$ $(\varepsilon = \varepsilon(v) > 0)$

Consider $\tilde{\gamma} : (-\lambda\varepsilon, \lambda\varepsilon) \longrightarrow M$ where

$$\tilde{\gamma}(t) = \gamma(\lambda^{-1}t), \quad t \in (-\lambda\varepsilon, \lambda\varepsilon).$$

- $\tilde{\gamma}'(0) = \lambda^{-1}\gamma'(0) = \lambda^{-1}v.$

- if $\lambda \gg 1$ s.t. $\lambda\varepsilon > 1$, then

  $\Rightarrow$ $\exp_p(\lambda^{-1}v)$ is well-defined in M.

$\therefore$ For the given $v \in T_pM$ with $|v| = 1$.

$\exp_p(tv)$ is well-defined $\forall\ 0 < t \ll 1$.

By stability of ODE, $\exists\, \varepsilon > 0$ s.t. $\forall\, 0 < t < \varepsilon$,

$\forall\, v \in T_pM$ w/ $|v| = 1$, $\exp_p(tv)$ is well-defined.

By passing $t \to 0$, $\exp_p$ is well-defined on a small nbd of origin.

Prop: $\exists\, \varepsilon_0 > 0$ s.t. $\exp_p : B(\varepsilon) \to M$ is a local diffeomorphism.

$B(\varepsilon) = \{v \in T_pM : |v| = \varepsilon\}$

pf: consider

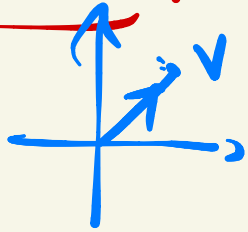
$$d\exp_p\big|_0 : T_0(T_pM) \cong \mathbb{R}^n \to T_pM.$$

$\exp_p(tv) = \gamma(t)$ where $\gamma(s)$ is the geodesic from $p = \gamma(0)$ with $\gamma'(0) = v$.

$\Rightarrow \exp(0) = \lim_{t \to 0} \gamma(t) = p.$

For $v \in T_0(T_pM) \cong T_pM$

$$d\exp_p\big|_0(v) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv)$$



$$= \frac{d}{dt}\Big|_{t=0} \gamma(t)$$ where $\gamma(t)$ is the geod. from p w/ $\gamma'(0)=v$

$$= \gamma'(0)$$

$$= v$$

$$\therefore \boxed{d\exp_p|_0 = Id} \neq \text{singular}$$

$$\Downarrow$$

Result by Implicit fcn thm.

<u>Rmk</u>: If $\exp_p : \underline{T_pM} \to M$ is diff.

then $M \cong \mathbb{R}^n$.

this is the case when curvature < 0.

prop $\Rightarrow$ exp map is a local coordination.

"$B_\varepsilon(0)$ to small Ball around p"

$$(x^1, x^2, \cdots, x^n) \in \mathbb{R}^n \longmapsto \exp_p\left(\sum_{i=1}^{n} x^i e_i\right) \in M$$

where $\{e_i\}$ = o.n. of T_pM.

$\forall\ v \in \mathbb{R}^n, |v|=1$, $\gamma(t) = \exp_p(tv)$ is geod. on M.

$\Rightarrow \quad \nabla_{\gamma'}\gamma' = 0 \quad \forall\ v \in T_pM,\ |v|=1.$

$\Rightarrow$ at p, taking $v = (0, \cdots, \overset{i\text{th}}{1}, 0 \cdots, \overset{j\text{th}}{1}, \cdots 0)$

we have $\nabla_{\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^j}\right)}\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial x^j}\right) = 0$ at p.

$$\Downarrow$$

torsion free

$$\cancelto{0}{\nabla_{\partial_i}\partial_i} + \cancelto{0}{\nabla_{\partial_j}\partial_j} + \nabla_{\partial_i}\partial_j + \nabla_{\partial_j}\partial_i = 0$$

$\Rightarrow \quad \boxed{\nabla_{\partial_i}\partial_j = 0}$ at p, $\forall\ i, j \in \{1, \cdots, n\}$

call : normal coordinat at p !!